Competitive storage and commodity price in continuous time

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Abstract

This paper offers a time-continuous competitive storage model and provides an analysis of the model solution. The model is relevant to the mineral commodity markets with traders managing inventories and getting speculative profits. The model includes serially correlated shocks of net supply and an upper boundary on the storage capacity. The no-arbitrage conditions on commodity trade imply the existence of the equilibrium price function under the standard boundary conditions on speculative trade. The equilibrium price is determined by the state variable defined as the long-term availability of commodity, which is the sum of storage and the expected cumulative disturbances of net supply. An approximate solution for a low-elastic net demand on commodity is derived in the explicit form. Numerical simulations of the equilibrium price function are conducted to examine the effects of the model parameters on this function.

Key words: commodity price, storage capacity, long-term availability, no-arbitrage condition, no-trade conditions, equilibrium price function

JEL Classification: Q02, C61, D84

1. Introduction

The competitive storage model is a well-established theoretical tool for studying the price behaviour of storable commodities. The model actors, apart from commodity producers and consumers, are competitive risk-neutral traders buying and selling a commodity to obtain

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speculative profits. Intertemporal storage arbitrage resulting from such trade generates the model price behaviour consistent with the observed qualitative features of commodity prices. These features include positive serial correlation, skewness of distribution, succession of long periods of doldrums and short periods of high prices. Positive autocorrelation of prices is explained by the smoothing effect of storage arbitrage that can buffer supply and demand shocks.

Beginning from Gustafson's (1958) major contribution to the study of the influence of storage on commodity price volatility, competitive storage models are formulated in discrete time. This is justified by the focus of empirical research based on this model on the prices of agricultural commodities. In the base model of commodity storage, production is exogenous and given by annual stochastic harvests that are serially uncorrelated. The annual price of agricultural commodity is formed after the realization of harvest, and decisions on how much to store of the available commodity are based on the annual price observed by traders.

However, solutions of the discrete-time storage models involve analytical difficulties caused by a non-negativity constraint on storage. Traders cannot borrow a commodity from future harvests, and stock-outs occur in the model under prices above a threshold level. The stationary rational expectations equilibrium is defined by a no-arbitrage condition implying a mapping of the current price into the next-period price expectation. Because of the non-negativity constraint, such a mapping implies a highly non-linear functional equation on price that cannot be linearized and solved analytically. Gustafson (1958) proposed a numerical method for solving the storage model. Since then, the sophisticated computational techniques have been developed for estimating the model parameters and numerical solutions by Wright (1991), Deaton and Laroque (1992, 1995, 1996), Cafiero et al. (2011), Guerra et al. (2015), Oglend and Kleppe (2017) and other authors.

This paper offers a different approach to the formal analysis of the competitive storage model and considers its modification for continuous time. The motivation is twofold. First, applying stochastic calculus under the assumption of a continuous-time stochastic process on supply or demand side facilitates formal analysis due to the possibility of linearization of the expected price growth differential. The second motive of this research is the relevance of the continuous-time model of competitive storage to mineral commodity prices, for which the data series with very short time periods can be a subject of empirical studies. For example, the daily data series are available for the West Texas Intermediate (WTI) crude oil prices. A time-continuous storage model can be applied for an empirical analysis of the high-frequency data series of commodity prices.

To some extent, the model of this paper rests upon the extensions of the base discrete-time model of annual commodity storage with serially uncorrelated supply shocks and unlimited storage capacity. An important extension was the introduction of positive serial correlation of supply or demand shocks into the base model. Chambers and Baily (1996) generalized the theoretical results of Deaton and Laroque (1992) for the base model by demonstrating the existence of a unique stationary rational expectations equilibrium for a model with the autocorrelated supply process. Deaton and Laroque (1995, 1996) showed that the price autocorrelations, implied by their econometric estimation of the model, are significantly lower than those observed for the series of actual commodity prices. Guerera et al. (2011, 2015) rejected this inference by using a finer approximation of the model solution. Dvir and Rogoff (2009) applied a discrete-time storage model with serially correlated and persistent demand shocks for the analysis of the crude oil price volatility.

Oglend and Kleppe (2017) extended the base storage model by introducing a storage capacity constraint to capture negative price spikes that can occur under supply gluts. They assumed that speculative storage is bounded from above and demonstrated that the sequence of price functions obtained by the fixed-point iteration of the price mapping converges to the equilibrium price function.

The model of this paper features both a positive intertemporal dependence of net supply disturbances, represented by a mean-reverting stochastic process, and a capacity constraint on storage, in addition to the non-negativity constraint. The model with unlimited storage capacity is considered as a special case.

The intertemporal equilibrium of the model is defined by the zero-profit condition of no-arbitrage for the interior solution and by the negative-profit conditions of no-trade for the corner solutions. Under *stock-out*, the non-negativity constraint on storage is binding, and the no-trade condition requires the expected profit from buying a commodity to be negative. Under *full storage capacity*, the capacity constraint on storage is binding and the no-trade condition is that the expected profit from selling a commodity is negative. The no-trade conditions ensure that traders have no incentives to buy under stock-out or to sell under full storage capacity.

The solution for a stationary rational expectations equilibrium of the model is given by an equilibrium price function of a state variable. In the continuous-time competitive storage model, the state variable is *the long-term availability* defined as the sum of storage and the expected cumulative net supply of commodity. Note that in the base discrete-time model with serially independent supply shocks, the state variable is *the current availability* defined as storage by the beginning of year plus annual harvest (e.g. Deaton and Laroque 1992). In the discrete-time models with serially correlated supply shocks, there are two state variables: storage and annual harvest (e.g. Chambers and Bailey 1996).

For the continuous-time model, the zero-profit condition of no-arbitrage implies, due to linearization, a second-order non-linear ordinary differential equation for the equilibrium price function. I demonstrate the existence and uniqueness of this function under the boundary conditions of value matching and smooth pasting. By using an approximation for this equation for a low-elastic net demand, I derive an explicit solution satisfying the boundary conditions. The equilibrium price function is given by a linear-quadratic combination of exponential functions of the long-term availability, which is derived for a bounded storage capacity and for the unbounded one as the limit case.

For a bounded storage capacity, there exist two types of equilibrium price function. For the first type, the regime switching between speculative trade and full storage capacity with no such trade occurs at a positive price. For the second type, this switching takes place at zero price coinciding with the kink point of a piecewise-linear inverse net demand function assumed in the model.

The second type of equilibrium solution is relevant to the recent empirical evidence on negative commodity price spikes. The crude oil oversupply shocks that occurred in the second quarter of 2020 provide an example of the influence of bounded storage capacity on the crude oil price dynamics. In April 2020, commercial storage in the United States approached to the level of full utilization, while the WTI crude price dropped to the zero level.

With the analytical solution of the model, a series of numerical simulations were conducted for the two types of capacity-constrained solutions and for the case of unbounded storage capacity. The equilibrium price functions are calculated under various values of the model parameters. In particular, simulations demonstrate the effects of storage capacity and the variance of net supply disturbances on the commodity price level and on the boundaries of the interval of speculative trade.

2. The model

Consumers and commodity producers are represented as a homogeneous group of market participants that do not hold storage but generate a net demand covered by competitive traders who own storage and get speculative profits from commodity trading. The net demand results from the joint reaction of producers and consumers to price movements and from random disturbances of the market. We consider, first, a commodity market without speculative traders and then introduce these participants.

2.1 Commodity market

The instantaneous net demand in the commodity market is represented as the difference between a price-dependent non-stochastic term called *the net demand* function $y(p)$, where p is the commodity price, and an exogenous random variable x that defines instantaneous *net supply* disturbances. Here and henceforth, we do not use a time variable in the model notations if it is not necessary.

The instantaneous net demand function $y(p)$ is defined for all $p \geq 0$, continuously differentiable, decreasing and convex. This function is positive for low prices and negative for high ones. An example of this function that will be used in what follows is the linear net demand shown in Figure 8.2a: $y(p) = b - \delta p$, with $\delta > 0$. The linear net demand function can represent the difference between inelastic demand $D(p) = b_d$ and linear supply $(p) = b_s + \delta p$, so that $b = b_d - b_s > 0$. The magnitude of parameter δ is small for a low-elastic supply.

Figure 2a Net demand function

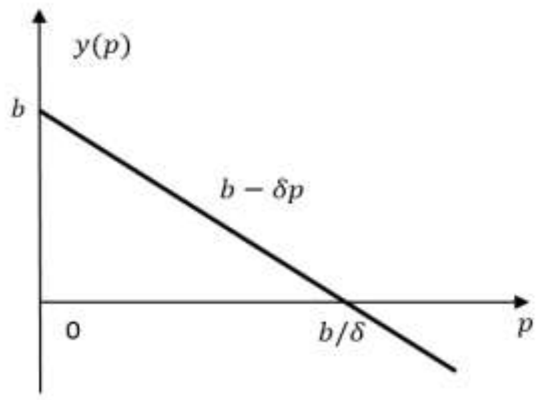
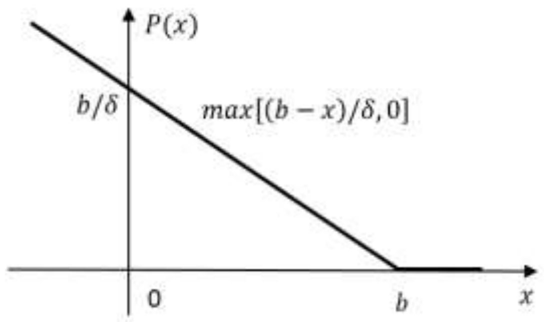


Figure 2b Market-clearing price



The net supply disturbances x follow the mean-reverting process:

$$dx = -\mu x dt + \sigma dw \ , \tag{1}$$

where $\mu > 0$ is the rate of mean reversion, dw is an increment of the standard Wiener process with the long-term mean 0 and instantaneous variance 1, and σ is the standard deviation of net supply fluctuations. The variance of these fluctuations depends linearly on the length of an infinitesimal time interval, $E(\sigma dw)^2 = \sigma^2 dt$. The stochastic equation (1) means that in any such interval $(t, t + dt)$, any small change of net supply dx results from the drift to the long-term zero mean, $-\mu x dt$, and the impact of serially uncorrelated shocks, σdw.

In the absence of speculative trade, the market clears at any instant, implying that

$$y(p) = x \; . \tag{2}$$

The market-clearing price satisfying (2) is the inverse net demand function of random shocks denoted as $P(x) \equiv y^{-1}(x)$. For the linear net demand function $y(p) = b - \delta p$, the market-clearing price is $P(x) = (b - x)/\delta$, if $x \leq b$. The price is zero under oversupply: $P(x) = 0$, if $x > b$. Hence, the market price in the absence of storage is the piecewise-linear function of net supply disturbance:

$$P(x) = max[(b - x)/\delta, 0]. \tag{3}$$

The function $P(x)$ is shown in Figure 2*b*. It is defined for all real numbers and intersects the vertical axis at point $P(0) = b/\delta$, which is the market-clearing price for the long-term mean of net supply process (1). The market cannot be cleared under oversupply because the volume of commodity $x - b$ cannot be sold at a positive price. Supply at zero price, $P(x) = 0$, means that, in the absence of storage, commodities are removed from the market.

2.2 Traders and storage

Suppose that competitive homogenous traders are present in the commodity market and consider a continuous-time trader's problem. A risk-neutral trader maximizes the expected discounted cash flow from trade over an infinite time horizon:

$$E_0 \int_0^\infty e^{-rt} p(t) q(t) dt, \tag{4}$$

subject to the storage balance equation:

$$ds(t) = -q(t) dt \; , \tag{5}$$

and the two-sided constraints:

$$0 \leq s(t) \leq \bar{s} \,, \tag{6}$$

where r denotes the riskless real interest rate, s storage, q the intensity of trade and $\bar{s}$ the storage capacity. Storage is costless and does not deteriorate in time. At any instant, traders observe the current price p and choose the intensity of trade q. They are commodity sellers if $q > 0$ and buyers if $q < 0$. For a small time interval $(t, t + dt)$ the volume of trade is qdt and the cash flow is $pqdt$. According to (5), the change of storage ds equals the volume of trade, $-qdt$. According to (6), storage is constrained from below by zero and from above by storage capacity. The initial storage is $s(0) \geq 0$.

At any instant, traders clear the market:

$$q = y(p) - x \,. \tag{7}$$

Thus, equations of the model with traders include the trader's problem (4) – (6), the market-clearing condition (7) and equation (1) specifying the process for net supply disturbances.

3. The intertemporal equilibrium conditions

The state of commodity market is defined at any instant by the storage capacity constraints (6). Trade with the use of storage may be present or absent, implying the equilibrium conditions on the expected rate of price growth.

3.1 Speculative trade

Trade with the use of storage occurs, if storage capacity is available and both constraints in (6) are non-binding, $0 < s < \bar{s}$. The no-arbitrage condition for speculative trade implies that the expected growth rate of price is equal to the interest rate:

$$E_t dp = rpdt. \tag{8}$$

The expected net profit is zero and the equilibrium commodity price dynamic is governed by the stochastic Hotelling rule (8).

3.2 Stock-out

Under stock-out $s = 0$, and the no-trade condition is that the expected rate of price growth is below the interest rate:

$$E_t dp < rpdt \ . \tag{9}$$

On the one hand, traders cannot sell the commodity because the stocks are absent. On the other hand, they do not buy because the expected rate of price growth is below the interest rate (it is better to deposit money in bank than to take a long position in commodity). Since there is no trade under stock-out ($q = 0$) the price fully absorbs any net supply disturbance x:

$$p = P(x) \ . \tag{10}$$

For the linear net demand function, $y(p) = b - \delta p$, the price under stock-out is given by (3).

3.3 Full storage capacity

Under full storage capacity $s = \bar{s}$, and traders do not buy the commodity, because they have no available storage capacity. They do not sell only if the expected growth rate of price is above the interest rate:

$$E_t dp > rpdt \ , \tag{11}$$

given that the price is positive, or if the price is zero, $p = 0$. In the former case, the expected rate of return on storing exceeds the interest rate (it is better to store the commodity than to sell and deposit money in bank). In the latter case, the sale at zero price means a gift to consumers that does not make sense for traders. Thus, speculative trade is absent, $q = 0$, under condition (11) or zero price. The price under full storage capacity fully absorbs shocks, $p = P(x)$, and is given by (3) for the linear net demand.

4. Stationary rational expectations equilibrium

In Deaton and Laroque's (1992) discrete-time model, a current availability means the amount of commodity available in any period to consumers and traders and defined as the sum of storage and supply per current period. Here *a long-term availability* of commodity is defined for any instant as the sum of storage and *the cumulative net supply* expected for the long term:

$$a(t) = s(t) + E_t \int_0^\infty x(t+\tau) d\tau \ , \tag{12}$$

The term added to storage is the conditional expectation of net supply integrated over an infinite time horizon. For the stochastic process (1), the expected lagged net supply for any time moment $t + \tau$ is $E_t x(t+\tau) = e^{-\mu\tau}x(t)$. Integrating over time interval $0 \le \tau \le T$, we obtain: $E_t \int_0^T x(t+\tau)d\tau = x(t)\int_0^T e^{-\mu\tau}d\tau = x(t)(1 - e^{-\mu T})/\mu$. For $T = \infty$, this equals $x(t)/\mu$, and formula (12) is written as

$$a(t) = s(t) + x(t)/\mu \; . \tag{12'}$$

The initial long-term availability is given by $a(0) = s(0) + x(0)/\mu$, where $x(0)$ is the initial net supply. The variable $a(t)$ can be negative if the net supply disturbance $x(t)$ is negative.

Consider a stationary rational expectations equilibrium, for which the stochastic net supply process (1) is stationary. We will show that the long-term availability defined as $a = s + x/\mu$ is the state variable sufficient to determine the equilibrium price. For the stationary equilibrium, consider the equilibrium price function of the long-term availability defined for all real numbers and twice continuously differentiable. Under speculative trade, this function denoted as $p(a)$, satisfies the no-arbitrage equation (8) rewritten as:

$$Edp(a) = rp(a)dt \; . \tag{13}$$

The expectation of price change for the stationary equilibrium is taken with respect to the state variable a that must contain all information relevant to the current price determination.

Under stock-out $a = x/\mu$. Let us introduce, for the sake of notational convenience, the inverse net demand as the function of availability:

$$\overline{P}(a) \equiv P(\mu a)) = max[(b - \mu a)/\delta, 0] \; .$$

The stock-out price is always positive, hence $\overline{P}(a) = (b - \mu a)/\delta$. Under full storage capacity $a = \bar{s} + x/\mu$, and the inverse net demand is represented as:

$$\underline{P}(a) \equiv P\big(\mu(a - \bar{s})\big) = max\big[\big(b + \mu(\bar{s} - a)\big)/\delta, 0\big] \; .$$

A *synthetic* equilibrium price function $\mathcal{P}(a)$ is defined for all real numbers by combining the intertemporal equilibrium conditions (8) through (11) as:

$$\mathcal{P}(a) = \begin{cases} \overline{P}(a)\,, & Ed\overline{P}(a) < r\overline{P}(a)dt \\ p(a)\,, & Edp(a) = rp(a)dt \\ \underline{P}(a) > 0, & Ed\underline{P}(a) > r\underline{P}(a)dt \\ \underline{P}(a) = 0 & \end{cases} \tag{14}$$

Conditions of storage balance (5) and market clearing (7) are represented as:

$$ds = xdt - y(\mathcal{P}(a))dt \; , \tag{15}$$

implying that for any small time interval the change of storage covers the change of net supply and net demand.

4.1 The equilibrium price function

Under trade, the state variable a is driven by the stochastic Ito process:

$$da = -y\big(p(a)\big)dt + (\sigma/\mu)dw \tag{16}$$

that results from combining the net supply process (1) with storage balance under market clearing (15): $da = ds + dx/\mu = xdt - y\big(p(a)\big)dt + (-\mu xdt + \sigma dw)/\mu = -y\big(p(a)\big)dt + (\sigma/\mu)dw$. The process for long-term availability (16) has the non-linear drift rate $-y\big(p(a)\big)$ and the instantaneous standard deviation σ/μ.

The standard deviation of the stochastic term in (16) is the ratio of σ to the rate of mean reversion μ. This is the standard deviation of the stochastic process for x *adjusted for autocorrelation*. The lower the coefficient μ, the higher the autocorrelation of x, and the higher is σ/μ, the standard deviation for a. If μ is small, the net supply disturbances are persistent and the instantaneous variance of long-term availability is large, as well as the variance of long-term net supply (for the mean-reverting process (1) the long-term variance is $\lim_{\tau\to\infty} Var_t x(t+\tau) = \sigma^2/2\mu$).

Under stock-out, $a = x/\mu$ and $da = dx/\mu$. The differential for the state variable is:

$$da = -y\big(\mathcal{P}(a)\big)dt + (\sigma/\mu)dw\ , \tag{17}$$

because, from (1), $da = dx/\mu = -xdt + (\sigma/\mu)dw$ and, from (15), $x = y\big(\mathcal{P}(a)\big)$ for $ds = 0$. Similarly, under full capacity, $a = \bar{s} + x/\mu$ and $da = dx/\mu = -xdt + (\sigma/\mu)dw$, hence the differential for the state variable is given by (17), because $x = y\big(\mathcal{P}(a)\big)$ for $ds = 0$.

It is important that the net supply term x vanishes from the right-hand side of equation (16) and does not influence directly the differential da in this equation. This is the consequence of our choice of availability measure as $a = s + x/\mu$. The stochastic process for availability (16) or (17) incorporates conditions of equilibrium: the storage balance (5) and the market clearing condition (7) for net supply disturbance x and net demand $y(p)$. The long-term availability a is indeed the state variable containing all information sufficient for the formation of price expectations and equilibrium price determination according to the intertemporal equilibrium condition (13).

Applying Ito's Lemma for the stochastic process for long-term availability (16), one can express the expected price differential as the first-order series expansion involving the second-order derivative of the equilibrium price function (EPF):

$$Edp(a) = -p'(a)y\big(p(a)\big)dt + \frac{1}{2}(\sigma/\mu)^2 p''(a)dt \; . \tag{18}$$

The expected price change results from the effect of net demand and the effect of the long-term availability variance.

Substituting the no-arbitrage condition (13) for the expectation term in equation (18), and dividing both sides by dt, yields the second-order non-linear differential equation for $p(a)$:

$$\frac{1}{2}(\sigma/\mu)^2 p''(a) - p'(a)y\big(p(a)\big) = rp(a) \; . \tag{19}$$

The EPF $p(a)$ is the solution of this equation subject to the boundary conditions defined below. The non-linear term on the left-hand side of (19) captures the feedback effect of net demand on the price change, which is taken into account by commodity traders in their expectations of price change.

Thus, at any instant, the equilibrium price $\mathcal{P}(a)$ in the continuous-time model is determined by the state variable a that evolves according to stochastic differential equation (16) or (17). The change of equilibrium storage is found at any instant as $ds = [x - y(\mathcal{P}(a))]dt$, according to (15). Consequently, the information about instantaneous net supply x is relevant to the change of storage, but it is redundant *per se* for determination of the EPF $p(a)$.

4.2 Equilibrium price paths

Consider the second-order differential equation for the EPF under trade (19) for the linear net demand $y(p) = b - \delta p$. One can represent this equation as the two-dimensional system of first-order differential equations for the EPF and its derivative:

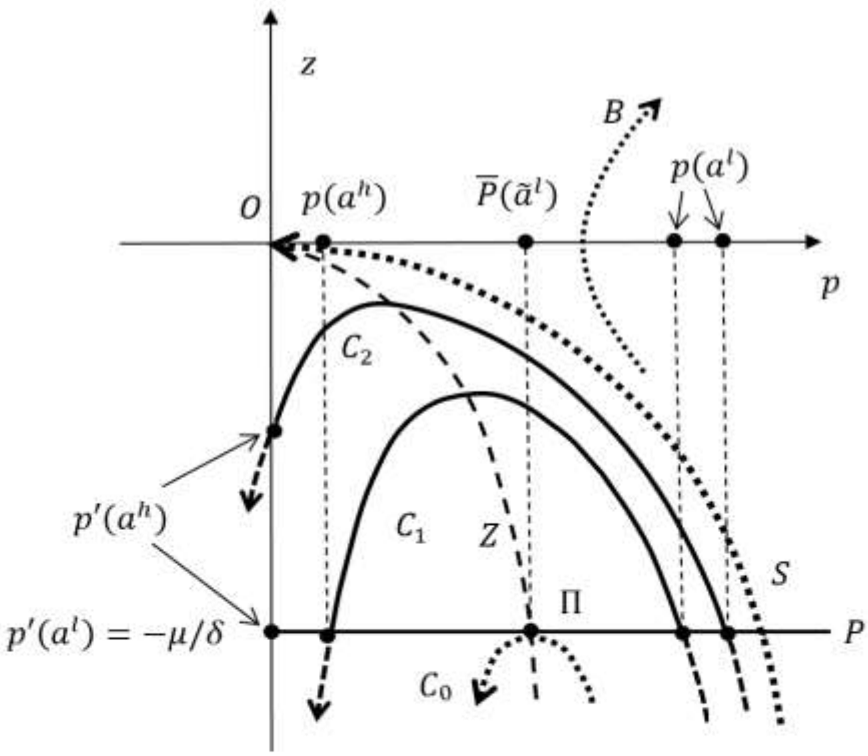
$$p'(a) = z(a) \tag{20}$$

$$\frac{1}{2}(\sigma/\mu)^2 z'(a) = z(a)\big(b - \delta p(a)\big) + rp(a) \; . \tag{21}$$

Equation (21) presents variable $z(a)$ as the derivative of equilibrium price with respect to the state variable, and equation (21) is identical to (19). A pair of price function $p(a)$ and its derivative $z(a)$ are the phase variables satisfying the system (20), (21).

Figure 4 depicts the phase plane of this system. Curved arrows show paths of price $p(a)$ and price derivative $z(a)$ under increasing argument a. We search for an EPF $p(a)$ and focus attention on the fourth quadrant of the phase plane, where the price is positive and the price derivative is negative, since the equilibrium price function we are looking for should be decreasing.

Figure 4 The phase plane and the equilibrium price paths



The stationary state of system (20), (21) is the origin O, because it is the intersection of the two loci, corresponding to zero derivatives: $p'(a) = 0$ and $z'(a) = 0$. The first one is line $z = 0$ and the second one is curve Z, the locus of zero second-order derivatives, shown in figure 4 and given by the function:

$$Z(p) = -\frac{rp}{b - \delta p} \ .$$

The characteristic equation of system linearized at the origin is

$$\begin{vmatrix} -\lambda & 1 \\ gr & gb-\lambda \end{vmatrix} = \lambda^2 - gb\lambda - gr = 0\,,$$

where λ is the characteristic root, $g = 2(\mu/\sigma)^2$. This equation has two real roots of a different sign, hence the stationary state is the saddle point.

As one can see in figure 4, there are several types of trajectories in the phase plane of the system (20), (21). The saddle path S is drawn with the bold dotted curved arrow in this figure. It converges to the origin O as a tends to infinity and it is the unique path in the fourth quadrant converging to the origin. The path, drawn with the dotted curved arrow B, is characterized by a non-monotonic price dynamic. The price is decreasing for $z < 0$ and, after intersecting the horizontal axis, $z = 0$, is increasing for $z > 0$. The price growth under increasing availability is caused by a speculative bubble generated by perpetual storage accumulation, which is driven by expectations of further price growth. Thus, each B-type path in Figure 4 represents a bubble solution.

We will focus on the types of trajectories C_1 and C_2 drawn in Figure 4 with the bold curved arrows. These are paths with a decreasing price (under increasing a) and non-monotonic price derivative. As any type-C_1 or C_2 path crosses curve Z (the locus of zero second-order derivatives, $z'(a) = 0$), the price derivative z starts to decrease, and the price falls thereafter at an accelerating pace.

Horizontal line P in Figure 4 depicts the locus of non-zero inverse net demand $\overline{P}(a)$ or $\underline{P}(a)$ in the phase plane. Each point of this locus is a combination of $\overline{P}(a)$, $\overline{P}'(a)$ or $\underline{P}(a)$, $\underline{P}'(a)$ for any a. For the linear net demand, locus P is given by the line:

$$z = -\mu/\delta\,,$$

because $\overline{P}'(a) = \underline{P}'(a) = -\mu/\delta$ under positive prices. The path of type C_1 intersects the line of inverse net demand P twice, while the path of type C_2 only once.

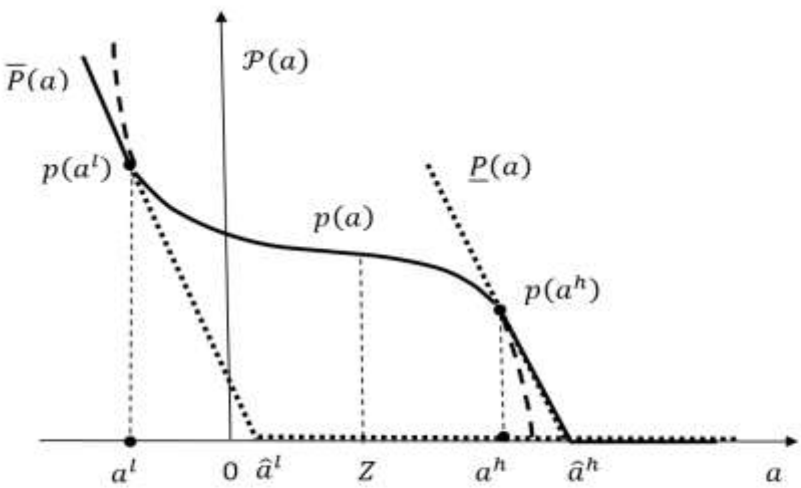
The price paths of both types represent the EPFs under a limited storage capacity $\bar{s} < \infty$. As will be shown below, a price path belongs to type C_1 if $\bar{s}$ is relatively small, and to type C_2 if it is relatively large. The saddle path S is the limit case for type C_2 corresponding to the unbounded storage capacity, $\bar{s} = \infty$.

4.3 The boundary conditions

The points of intersection of the price paths of types C_1 and C_2 with line P are depicted in figure 4 as $\left(p(a^l), p'(a^l)\right)$ and $\left(p(a^h), p'(a^h)\right)$. These are the points of switching between the regimes of stock-out, trading and full storage capacity that define the structure of the synthetic EPF $\mathcal{P}(a)$.

Figure 5 portrays this function for the price path of type C_1, It is drawn with the bold solid curve and consists of three pieces. The first one, the EPF under trading $p(a)$ is a solution of the second-order differential equation (20)-(21). The function $p(a)$ is decreasing and convex-concave with the inflection point Z, because derivative $z(a)$ for the price path of type C_1 depicted in figure 4 increases with a before the intersection of the locus of second-order derivatives and then $z(a)$ decreases. The function $p(a)$ in figure 5 touches tangentially two other pieces of $\mathcal{P}(a)$, the piecewise-linear functions of inverse net demand: $\overline{P}(a)$ under stock-out and $\underline{P}(a)$ under full storage capacity, which are drawn as the kinked dotted lines.

Figure 5 The equilibrium price function of type C_1



As shown in figure 5, the values and the derivatives of functions $p(a)$ and $\overline{P}(a)$ coincide at the switching point a^l. Similarly, the values and the derivatives of $p(a)$ and $\underline{P}(a)$ coincide at the switching point $a^h > a^l$.

In other words, the switching points a^l and a^h satisfy the conditions of value matching and smooth pasting. These conditions establish a connection between the unknown

EPF $p(a)$ and the known inverse net demand functions $\overline{P}(a)$ and $\underline{P}(a)$. The value-matching condition requires the continuity of $\mathcal{P}(a)$ at points a^l and a^h:

$$p(a^l) = \overline{P}(a^l), \qquad p(a^h) = \underline{P}(a^h)\,. \tag{22}$$

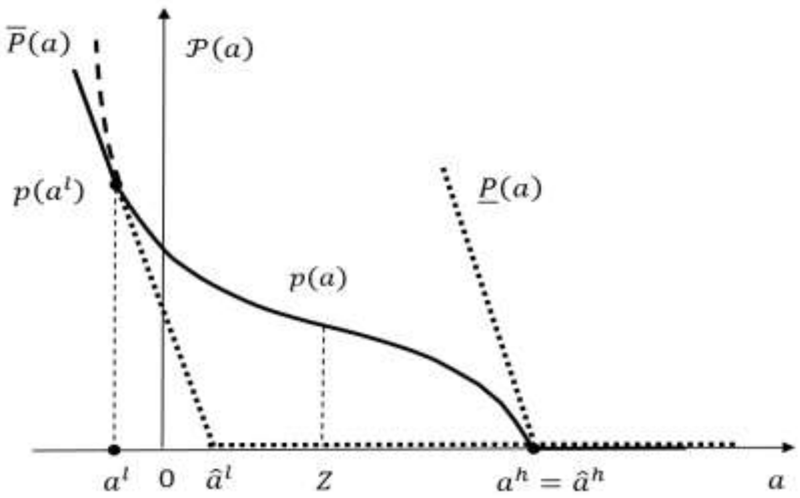
At these points, traders are indifferent between trading activity and inaction. The smooth-pasting condition ensures the absence of arbitrage at the switching points and means that $\mathcal{P}(a)$ is differentiable at these points:

$$p'(a^l) = \overline{P}'(a^l), \qquad p'(a^h) = \underline{P}'(a^h)\,. \tag{23}$$

As one can see in figure 5, both switching points are below the kink points of inverse net demand functions $\overline{P}(a)$ and $\underline{P}(a)$: $a^l < \hat{a}^l$, $a^h < \hat{a}^h$, where $\hat{a}^l = b/\mu$ is the kink point for $\overline{P}(a)$ and $\hat{a}^h = \overline{s} + b/\mu$ is the kink point for $\underline{P}(a)$.

Figures 6 portrays the synthetic EPF $\mathcal{P}(a)$ for an equilibrium price path of type C_2. The EPF under trading $p(a)$ is decreasing and convex-concave. The values and derivatives of functions $p(a)$ and $\overline{P}(a)$ coincide at the lower switching point a^l. The function $p(a)$ intersects $\underline{P}(a)$ at the upper switching point a^h, which is the kink point of $\underline{P}(a)$, $a^h = \hat{a}^h$.

Figure 6 The equilibrium price function of type C_2



For the EPF of type C_2, the value-matching conditions (22) are fulfilled for both switching points a^l and a^h. However, the smooth-pasting condition is fulfilled only at the lower switching point:

$$p'(a^l) = \overline{P}'(a^l)\ , \tag{24}$$

whereas the derivative at point a^h varies between $-\mu/\delta$ and 0, as one can see in figure 4.

Thus, the structure of the EPFs demonstrated in figures 5, 6 is defined by the boundary conditions (22), (23) for type C_1 or (22), (24) for type C_2. The stock-out occurs at high prices that are above the upper boundary $p(a^l)$ and $\mathcal{P}(a) = \overline{P}(a)$ for $a < a^l$. The full storage capacity takes place at low prices that are below the lower boundary $p(a^h)$ and $\mathcal{P}(a) = \underline{P}(a)$ for $a > a^h$. Under trading $\mathcal{P}(a) = p(a)$ and $\overline{P}(a) \leq p(a) \leq \underline{P}(a)$ for $a^l \leq a \leq a^h$. The dashed curves in figures 5, 6 depict the off-equilibrium continuations of the price function $p(a)$ in the zones of stock-out and full storage capacity. For these continuations, the zero-profit condition $Edp(a) = rp(a)dt$ is fulfilled.

Conditions of value matching and smooth pasting hold for both switching points for type C_1, but for type C_2 the smooth-pasting condition is fulfilled only for a^l, the lower switching point. It does not hold for the upper switching point a^h, but arbitrage is impossible at this point, because it coincides with the kink point of $\underline{P}(a)$, $a^h = \hat{a}^h$ (figure 6). At this point, traders have no place to store the commodity supplied at zero price. Otherwise, they would have obtained infinite returns by reselling it at a positive price.

4.4 Existence and uniqueness of equilibrium

There exists a unique equilibrium price path for any storage capacity varying between zero and infinity. One can see in figure 4 a correspondence between the size of this capacity and the equilibrium price path.

For zero storage capacity the price path satisfying the no-arbitrage condition (19) is tangent to the horizontal line of inverse net demand P. This path is drawn with the dotted curved arrow C_0 in figure 4. The point of tangency Π in the figure coincides with the point of intersection of line P and curve Z, the locus of zero second-order derivatives. It is shown in Appendix A that $\Pi = \left(\overline{P}(\tilde{a}^l\ ), -\mu/\delta\right)$, where

$$\tilde{a}^l\ = \frac{r(b/\mu)}{r+\mu} = \frac{r\hat{a}^l}{r+\mu}$$

is the solution of equation: $Z\left(\overline{P}(a)\ \right) = -\mu/\delta$, and the zero-profit no-arbitrage condition (19) is fulfilled in the absence of speculative trade (for $\bar{s} = 0$) only at price $\overline{P}(\tilde{a})$.

For any storage capacity $\bar{s}$ above zero, the equilibrium price path intersects curve Z at one point and line P at two points a^l and a^h that define the interval of speculative trade, so that

$$a^l < \tilde{a}^l < a^h, \tag{25}$$

as one can see in figure 4. If storage capacity is sufficiently small, the equilibrium price path corresponding to $\bar{s}$ is of type C_1. With an increase of this capacity, the trading zone widens and the equilibrium price path transforms to type C_2. For storage capacity tending to infinity, the price paths of type C_2 converge to the saddle path S.

4.5 The no-trade conditions

The condition of no-trade under stock-out in (14) is given by inequality: $Ed\overline{P}(a) < r\overline{P}(a)dt$. It is shown in Appendix B that in the stock-out zone, $a < a^l$, the expected growth rate is lower for the inverse net demand $\overline{P}(a)$ than for an off-equilibrium continuation of the EPF under trade given by the price function $p(a')$:

$$\frac{Ed\overline{P}(a)}{\overline{P}(a)} < rdt = \frac{Edp(a')}{p(a')}\,, \tag{26}$$

where $a' = s + x/\mu < a^l$ and $s < 0$. Hence, the no-trade condition is fulfilled for $a < a^l$.

Similarly (see Appendix B), the no-trade condition under full storage capacity in (14), $Ed\underline{P}(a) > r\underline{P}(a)dt$, holds for $a^h < a < \hat{a}^h$ (see figure 5), because the expected growth rate is higher for $\underline{P}(a)$ than for an off-equilibrium continuation of $p(a)$ in this zone:

$$\frac{Ed\underline{P}(a)}{\underline{P}(a)} > rdt = \frac{Edp(a')}{p(a')}, \tag{27}$$

where $a^h < a' = s + x/\mu < \hat{a}^h$ and $s > \bar{s}$. Condition (27) is relevant only to the EPFs of type C_1, because $a^h = \hat{a}^h$ for type C_2.

Thus, the expected growth rate of price in the absence of speculative trade is compared with the growth rate for the virtual price paths, for which it equals the interest rate. In the first case, the expected growth rate for $\overline{P}(a)$ is below this rate for an off-equilibrium continuation of the EPF (drawn on the top of figures 5 and 6 with the dashed curves). For this continuation, a virtual accumulation of negative “storage” (or “borrowing” commodity from the future) ensures the no-arbitrage equality in (26), but implies a decline of the long-term availability that contributes to the price growth. In the second case, the expected growth rate for $\underline{P}(a)$ is above this rate for an off-equilibrium continuation of the EPF of type C_1 (drawn at the bottom of figure 5 with the dashed curve). For this continuation, the virtual accumulation of storage above the capacity ensures the no-arbitrage equality in (27), but leads to a drop of price to zero.

There is no trade for the EPFs of both types under full storage capacity and zero price, that is for $a^h > \hat{a}^h$.

5. Solutions for a low-elastic net demand

One can assume that parameter δ of net demand function $y(p) = b - \delta p(a)$ is small to solve explicitly the system of non-linear differential equations (20)-(21). First, consider a solution for the linear homogenous equation, which is the case for $\delta = 0$:

$$\tfrac{1}{2}(\sigma/\mu)^2 p''(a) - bp'(a) - rp(a) = 0\ . \tag{28}$$

Substituting a partial solution $Ae^{\lambda a}$ into (28), where A and λ are unknown parameters, we obtain the quadratic characteristic equation on λ:

$$\tfrac{1}{2}(\sigma/\mu)^2 \lambda^2 - b\lambda - r = 0\ .$$

The two roots of this equation are

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 + 2(\sigma/\mu)^2 r}}{(\sigma/\mu)^2}$$

such that $\lambda_1 > 0$, $\lambda_2 < 0$. The general solution of equation (28) is a linear combination of partial solutions:

$$p^{(0)}(a) = A_1 e^{\lambda_1 a} + A_2 e^{\lambda_2 a}\ , \tag{29}$$

where A_1, A_2 are unknowns. One can interpret $p^{(0)}(a)$ as an EPF under inelastic net demand, when $\delta = 0$.

Now consider an approximate solution for the non-linear equation (21) rewritten as:

$$\tfrac{1}{2}(\sigma/\mu)^2 p''(a) - bp'(a) - rp(a) = -\delta p'(a)p(a)\ . \tag{30}$$

It is shown in Appendix C that the first-order approximation of the solution for this equation for a small parameter δ is given by:

$$p(a) = p^{(0)}(a) + \delta p^{(1)}(a)\ , \tag{31}$$

where

$$p^{(1)}(a) = \beta_1\left(A_1 e^{\lambda_1 a}\right)^2 + \beta_{12} A_1 A_2 e^{(\lambda_1+\lambda_2)a} + \beta_2\left(A_2 e^{\lambda_2 a}\right)^2, \tag{32}$$

$$\beta_1 = -2\left(b + 3\sqrt{b^2 + 2(\sigma/\mu)^2 r}\right)^{-1}\ , \qquad \beta_2 = 2\left(3\sqrt{b^2 + 2(\sigma/\mu)^2 r} - b\right)^{-1}\ ,$$

$$\beta_{12} = -\frac{2b}{r(\sigma/\mu)^2}\ .$$

The equilibrium price function (31) is the sum of the EPF under inelastic net demand, $p^{(0)}(a)$, ~~a~~nd the term $\delta p^{(1)}(a)$ related to the market reaction on the price change, which is taken into account by commodity traders. The latter term, as expressed by formula (32), is the quadratic form of the partial solutions of the linear homogenous equation (28).

For the obtained approximate solution given by formulas (31), (29), (32) one have to find two unknown parameters A_1, A_2 and two unknown switching points a^l, a^h. The restriction imposed on parameters A_1, A_2 is that the function $p(a)$ must be decreasing and convex-concave in the interval of trading $[a^l, a^h]$.

5.1 Equations for the model solution

Let us introduce the new variables:

$$y_{1l} = \delta A_1 e^{\lambda_1 a^l}\ ,\quad y_{2l} = \delta A_2 e^{\lambda_2 a^l} \tag{33}$$

$$y_{1h} = \delta A_1 e^{\lambda_1 a^h},\quad y_{2h} = \delta A_2 e^{\lambda_2 a^h}. \tag{34}$$

For an equilibrium price function of type C_1, conditions of value-matching and smooth-pasting (22), (23) can be expressed for these four variables and the switching points a^l, a^h as the six-dimensional system of non-linear equations:

$$y_{1l} + y_{2l} + \beta_1 y_{1l}^2 + \beta_{12} y_{1l} y_{2l} + \beta_2 y_{2l}^2 = b - \mu a^l \tag{35}$$

$$\lambda_1 y_{1l} + \lambda_2 y_{2l} + 2\lambda_1\beta_1 y_{1l}^2 + (\lambda_1+\lambda_2)\beta_{12} y_{1l} y_{2l} + 2\lambda_2\beta_2 y_{2l}^2 = -\mu \tag{36}$$

$$y_{1h} + y_{2h} + \beta_1 y_{1h}^2 + \beta_{12} y_{1h} y_{2h} + \beta_2 y_{2h}^2 = b + \mu(\bar{s} - a^h) \tag{37}$$

$$\lambda_1 y_{1h} + \lambda_2 y_{2h} + 2\lambda_1\beta_1 y_{1h}^2 + (\lambda_1 + \lambda_2)\beta_{12} y_{1h} y_{2h} + 2\lambda_2\beta_2 y_{2h}^2 = -\mu \quad (38)$$

$$lny_{1h} - lny_{1l} = \lambda_1(a^h - a^l) \quad (39)$$

$$lny_{2h} - lny_{2l} = \lambda_2(a^h - a^l)\ . \quad (40)$$

Equations (35)-(38) follow directly from the value-matching and smooth-pasting conditions (22), (23) applied for the EPF (31), and equations (39), (40) follow from (33), (34).

For an equilibrium price function of type C_2, the smooth-pasting condition does not hold for the upper switching point a^h. This is the kink point for the inverse net demand function $\underline{P}(a\ )$, such that $\underline{P}(a^h) = 0$, hence:

$$a^h = \bar{s} + b/\mu\ . \quad (41)$$

The value-matching condition (22) for this point implies $p(a^h) = 0$. Consequently, the boundary conditions for the EPF of this type are determined by (41) and the system of five non-linear equations for variables a^l, y_{il}, y_{ih}, $i = 1,2$:

$$y_{1l} + y_{2l} + \beta_1 y_{1l}^2 + \beta_{12} y_{1l} y_{2l} + \beta_2 y_{2l}^2 = b - \mu a^l \quad (42)$$

$$\lambda_1 y_{1l} + \lambda_2 y_{2l} + 2\lambda_1\beta_1 y_{1l}^2 + (\lambda_1 + \lambda_2)\beta_{12} y_{1l} y_{2l} + 2\lambda_2\beta_2 y_{2l}^2 = -\mu \quad (43)$$

$$y_{1h} + y_{2h} + \beta_1 y_{1h}^2 + \beta_{12} y_{1h} y_{2h} + \beta_2 y_{2h}^2 = 0\ , \quad (44)$$

and also two equations identical to (39), (40).

5.2 The limit case: the saddle-path solution

If the storage capacity $\bar{s}$ tends to infinity, then the upper boundary a^h for paths of type C_2 also tends to infinity due to (41). The boundary conditions are

$$p(a^l) = \overline{P}(a^l), \quad p'(a^l) = \overline{P}'(a^l) \quad (45)$$

and

$$\lim_{a^h\to\infty} p(a^h) = 0, \quad \lim_{a^h\to\infty} p'(a^h) = 0\ . \quad (46)$$

As one can see from the phase plane in figure 4, the limit boundary conditions (46) are fulfilled for the saddle path S, whereas the paths of type C_2 converge to the saddle path as $\bar{s}$ tends to infinity.

In the limit case, the capacity constraint $s \leq \bar{s}$ does not affect the price. A solution for the saddle path is the special case of the EPF (31), when $A_1 = 0$ and the terms with positive exponent $A_1 e^{\lambda_1 a}$ vanish:

$$p(a) = A_2 e^{\lambda_2 a} + \delta\beta_2 \left(A_2 e^{\lambda_2 a}\right)^2, \tag{31'}$$

From (33), condition $A_1 = 0$ implies that $y_{1l} = 0$. The boundary conditions (45) for the saddle-path solution can be represented as:

$$2\beta_2 y_{2l}^2 + y_{2l} + \mu/\lambda_2 = 0 \tag{47}$$

$$\mu a^l = b - y_{2l} - \beta_2 y_{2l}^2 \,. \tag{48}$$

These are equations (42), (43) rewritten for $y_{1l} = 0$. The equilibrium solution is defined by the two unknown variables: y_{2l}, a^l. Parameter $A_2 > 0$ is found from (33) as $A_2 = y_{2l}/(\delta e^{\lambda_2 a^l})$. The equilibrium exists and is unique, because the intercept in the square equation (47) is negative, since $\lambda_2 < 0$, and the solution is given by $y_{2l} > 0$. The system (47), (48) is solved explicitly: the positive root of (47) is inserted into equation (48) to determine the boundary a^l.

6. Numerical simulations

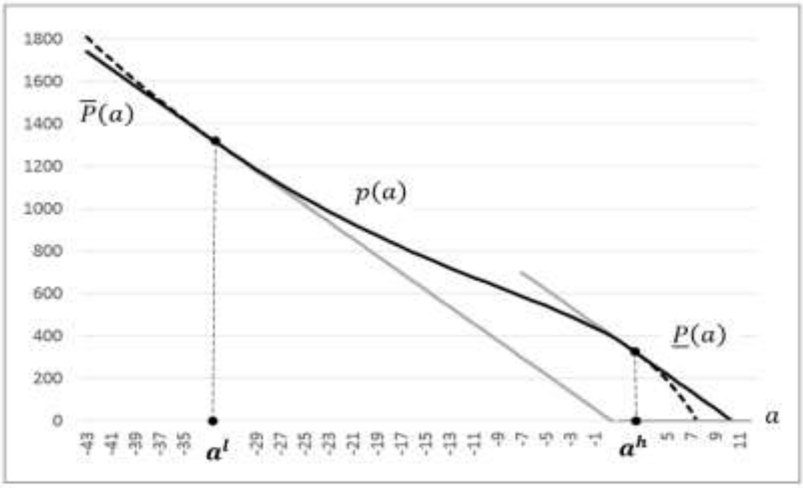
Now we can calculate the equilibrium price functions numerically for arbitrarily selected bundles of exogenous model parameters: r, b, δ, σ, μ, $\bar{s}$. For each bundle, we find, by solving the system of equations (35)-(40) or (39)-(44), the unknown parameters A_1, A_2 and the variables a^l, a^h that define the EPF under trading $p(a)$ and the synthetic price function $\mathcal{P}(a)$. These functions are calculated by the formulas (31), (29), (32) for the two types of equilibrium solution: C_1 and C_2. The results of simulations are presented graphically below.

6.1 Equilibrium price functions of type C_1

Consider a numerical example for the following values of the model parameters: $r = 0.015$, $b = 1$, $\delta = 0.05$, $\sigma = 6$, $\mu = 2$, $\bar{s} = 10$. The solution for the unknown variables a^l, a^h and parameters A_1, A_2 is found from the system (35)-(40), which is solved for six variables: a^l, a^h, y_{1l}, y_{1h}, y_{2l}, y_{2h}. Parameters A_1, A_2 are calculated from equalities (33), (34) that must hold as identities for any solution of this system.

The synthetic equilibrium price function $\mathcal{P}(a)$ is depicted in figure 7. It consists of three pieces pasted together: the price functions under trading $p(a)$, the inverse net demand functions under stock-out $\overline{P}(a)$ and under full storage capacity $\underline{P}(a)$. One can see from this figure that the value-matching and smooth-pasting conditions (22), (23) are satisfied at the switching points $a^l = -32.58$ and $a^h = 2.03$. The dashed curves show the off-equilibrium continuations of function $p(a)$ in the zones of stock-out, $a < a^l$, and full storage capacity, $a > a^h$.

Figure 7: The synthetic equilibrium price function $\mathcal{P}(a)$.



For the same numerical example we consider the interest rate change from $r = 0.015$ to $r = 0.02$, while leaving other parameters the same. Figure 8 demonstrates the effects of the interest rate increase on the EPF. The inverse net demand functions $\overline{P}(a)$ and $\underline{P}(a)$ drawn with grey lines do not alter, but the boundaries of trading zone $[a^l, a^h]$ change notably. This zone shifts to the right and narrows.

Figure 8: The equilibrium price functions for different interest rates

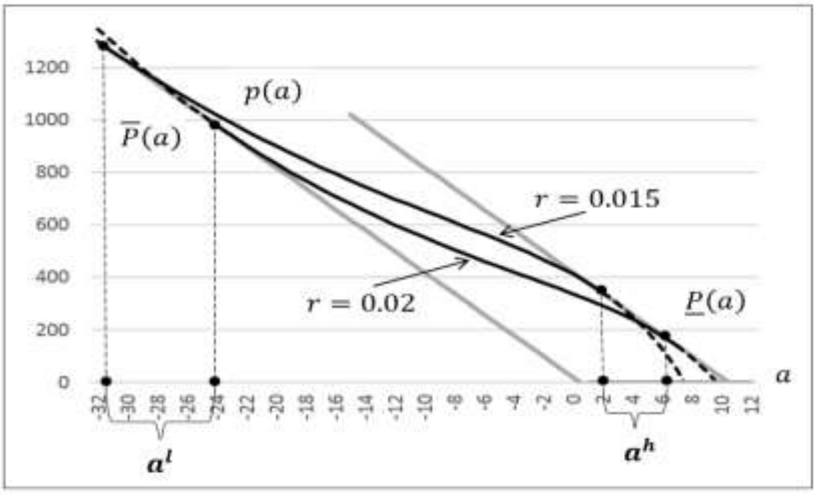


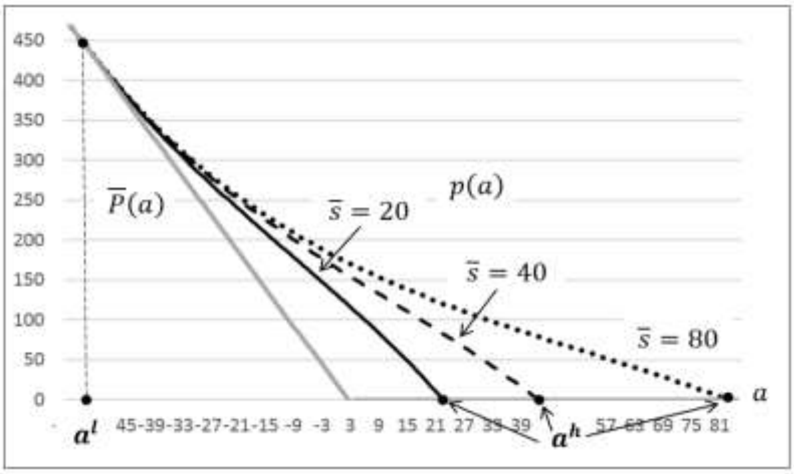
Figure 8 also demonstrates that a lower interest rate results in a higher level of equilibrium prices. A lower interest rate is more favourable for investment in commodity for the following reason. From the no-trade conditions in (14), buying the commodity under stock-out is unprofitable: $Ed\overline{P}(a) < r\overline{P}(a)dt$, but the zone $a < a^l$ satisfying this condition narrows for a lower interest rate. In contrast, a higher interest rate brings about a lower price level, because it is more favourable for sales of commodity by traders. The no-trade condition under full capacity, $Ed\overline{P}(a) > r\overline{P}(a)dt$, makes sales of commodity unprofitable. The zone $a > a^h$ satisfying this condition (for positive prices) narrows under a higher interest rate.

6.2 Equilibrium price functions of type C_2

Numerical simulations for the price function of type C_1 show that under a sufficiently large storage capacity $\overline{s}$ the price at the upper boundary of trading zone a^h becomes negative, $p(a^h) < 0$. In such a case, the solution of type C_1 should be ruled out, and one have to search for a solution of type C_2. It satisfies the system of equations (39)-(44) with the upper boundary a^h calculated as (41) and without condition of smooth pasting (38) for this boundary.

Consider a numerical example with parameter values: $r = 0.03$, $b = 1$, $\delta = 0.05$, $\sigma = 7$, $\mu = 0.4$. Figure 9 demonstrates the EPFs for different values of storage capacity: $\overline{s} = 20$, $40,\ 80$. All the lower boundaries a^l locate near the value $a = -54$, while the upper boundaries a^h, for which $p(a^h) = 0$, spread along the horizontal axis. One can see from figure 9 that a larger level of $\overline{s}$ implies a higher point of switching to the full-capacity regime.

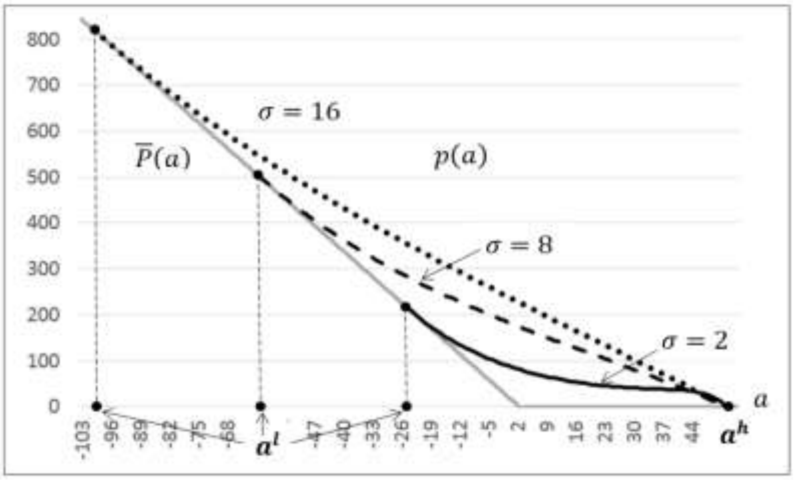
Figure 9: The EPFs for various storage capacities



The essential feature of the EPFs of type-C_2 is that the volume of storage capacity $\overline{s}$ is sufficient for the price to fall to zero, just when the full capacity is reached. For the EPFs of type-C_1, the full capacity is reached at a positive price, because of a lower volume of storage capacity. The other feature of the type-C_2 equilibrium is that a larger magnitude of storage capacity implies a higher level of prices, as one can see from figure 9. Traders can accumulate greater volumes of storage due to a larger amount of space available in their storehouses, and this is captured in the price expectations that have an upward effect on the commodity prices.

Figure 10 shows the EPFs for the different values of the standard deviation of net supply process: $\sigma = 2,\ 8,\ 16$, the storage capacity $\overline{s} = 50$, with other parameters the same as in the previous example. One can see from the figure that the upper switching point a^h is invariant for all functions $p(a)$, because a^h does not depend on σ, due to condition (41). The lower switching point a^l is increasing and the trading zone $[a^l, a^h]$ is widening with the standard deviation.

Figure 10: The EPFs for various standard deviations



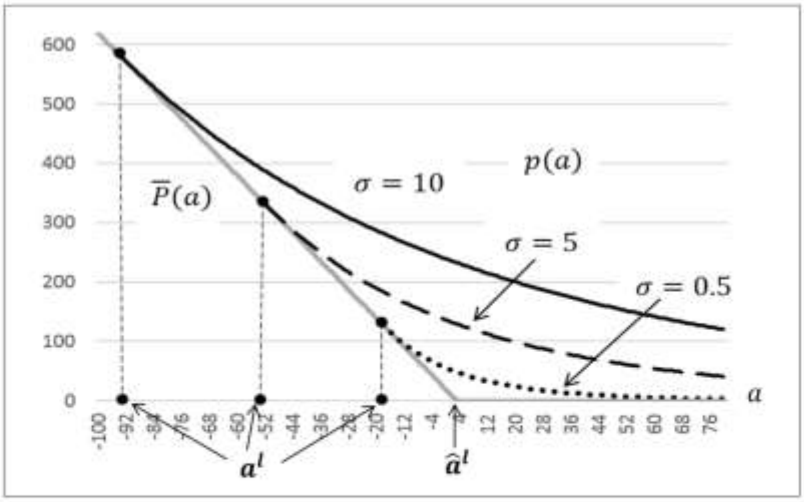
Thus, a higher volatility of net supply results in a widening of the trading zone. The inverse net demand $\overline{P}(a)$ does not change with the increase of σ, while the lower boundary a^l shifts to the left, as shown in figure 10. The threshold price $p(\ a^l) = \overline{P}(\ a^l)$, hence, increases with σ and causes an increase of the price level for all a in the trading zone. The intuitive reason for this effect is that traders have to hold larger volumes of storage under a higher volatility to ensure the fulfilment of no-arbitrage condition (19). In other words, a higher volatility of net supply stimulates demand for storage that leads to a general increase of the price level.

6.3 *Equilibrium price functions for the limit case* $\overline{\boldsymbol{s}} = \infty$

Finally, consider the case of unlimited storage capacity. The EPF under trading $p(a)$ for this case is given by formula $(31')$, and the solution for the synthetic price function $\mathcal{P}(a)$ is defined by equations (47), (48) for variables a^l and y_{2l}.

The EPFs depicted in figure 11 are calculated for the numerical example: $r = 0.03$, $b = 1$, $\delta = 0.05$, $\mu = 0.3$ and $\sigma = 0.5,\ 5,\ 10$. As in the previous example, the equilibrium price level is higher for a larger variance of net supply process.

Figure 11: The EPFs for various standard deviations, $\bar{s} = \infty$



This figure and the previous one illustrate a real-option nature of commodity prices. The real option valuation is a consequence of the dual nature of any commodity as a consumption good and a storable asset. Traders have an option to store a unit of commodity, and the option value of storing equals the difference between the equilibrium price and the inverse net demand: $v^{opt}(a) = p(a) - \bar{P}(a)$. As one can see from figure 11, the option value as a function of the state variable is increasing with the state variable volatility. Any option value function $v^{opt}(a)$ reaches maximum at the kink point $\hat{a}^l = b/\mu$, above which the commodity is worthless as a consumption good.

7. Conclusion

We have considered the model of competitive storage in continuous time with the serially correlated stochastic process of net supply and the bounded storage capacity. A continuous-time model of storage and trade is relevant to markets of mineral commodities, for which the high-frequency data on prices and commercial storage are available. The stochastic variable of net supply in the model relates to disturbances on demand and supply sides. The introduction of the upper-bound constraint on storage capacity allows the model to reflect phenomena of supply gluts and negative price spikes that happen sometimes in commodity markets.

The focus of this article was on the formal analysis of the theoretical model of competitive storage. The key point of the model solution is the choice of the state variable that determines the equilibrium price. For the continuous-time model, the state variable is the long-term availability defined as the sum of storage and the expected cumulative net supply disturbances. The latter term reflects expectations of traders with regard to future shocks on demand and supply sides affecting the commodity market.

For the stationary rational expectations equilibrium, we derived the non-linear differential equations for the equilibrium price function and demonstrated the existence and uniqueness of equilibrium solution. The standard boundary conditions determine the synthetic EPF and the interval of the state variable where speculative trade occurs. The approximation of the equilibrium solution for a low-elastic net demand makes possible to derive the explicit solution for price and to conduct the numerical calculations of the EPF for various combinations of the model parameters.

The goal of the calculations was to illustrate the properties of equilibrium solution and to analyze qualitatively a dependence of the EPF on the model parameters. The simulation results support the theoretical inferences about the shape of the EPF and reveal two types of equilibrium price functions under constrained storage capacity. It was shown, for example, that the price level is increasing and the size of trading interval is widening with a decrease of the interest rate and an increase of the storage capacity or the variance of net supply disturbances.

The model parameters were selected arbitrarily for the numerical calculations. It is possible to estimate parameters of the net demand function and the stochastic process of net supply for the real-world commodity markets by using the detrended price series and the data series of commercial storage. With the econometric estimates of parameters, one can calculate the excess supply shocks and the long-term availability series and then evaluate the EPFs for the functional form derived in this article.

It is also possible to relax the assumption of linear net demand function $y(p)$ that simplifies approximate solution and synthesis of the EPF. The model solution is essentially similar for a piecewise linear net demand, but may be more difficult when both supply and demand functions are of a more general type. In these cases, the inverse net demand function, which is the market-clearing price in the absence of speculative trade, is given implicitly. It has to be included in an implicit form into the equations for the model solution related to the boundary conditions. Such a generalization of the model can be a subject of further research.

Appendix A: *The point of tangency Π*

Consider the equation: $Z(p) = -\mu/\delta$, where $Z(p) = -rp/(b - \delta p)$ corresponds to curve Z in figure 4. The solution is given by

$$\tilde{p} = \frac{\mu(b/\delta)}{r + \mu}$$

The solution for equation $\tilde{p} = \overline{P}(a)$, where $\overline{P}(a) = (b - \mu a)/\delta$, is

$$\tilde{a}^l = \frac{r(b/\mu)}{r + \mu} = \frac{r\hat{a}^l}{r + \mu},$$

hence, $\tilde{a}^l$ is the solution of equation: $Z\left(\overline{P}(a)\right) = -\mu/\delta$. In the absence of speculative trade (for $\overline{s} = 0$), the zero-profit no-arbitrage condition (19) is fulfilled only at point $\tilde{a}^l$. Indeed, from (17), $Eda = -xdt = -\mu adt$, and the difference $Ed\overline{P}(a) - r\overline{P}(a)dt$ is linear and increasing in a:

$$Ed\overline{P}(a) - r\overline{P}(a)dt = \overline{P}'(a)Eda - r(b - \mu a)dt/\delta = (\mu/\delta)\mu adt + (\mu/\delta)radt - (rb/\delta)dt \;= \big((\mu/\delta)(\mu + r)a - (rb/\delta)\big)dt\ .$$

It equals zero only for $a = \tilde{a}^l$.

Appendix B: *Inequalities (26), (27)*

From the proof in Appendix A it follows that $Ed\overline{P}(a) < r\overline{P}(a)dt$ if, and only if, $a < \tilde{a}^l$. This inequality is fulfilled for $a < a^l$, because $a^l < \tilde{a}^l$ from (25) and the difference $Ed\overline{P}(a) - r\overline{P}(a)dt$ is increasing in a. As a result, inequality (26) is fulfilled for the off-equilibrium continuation of the EPF $p(a')$, where $a' = s + x/\mu < a^l$ and $s < 0$, because $Edp(a') = rp(a')dt$.

Consider inequality (27). Rearrange the difference:

$$Ed\underline{P}(a) - r\underline{P}(a)dt = \underline{P}'(a)Eda - r\big(b + \mu(\overline{s} - a)\big)dt/\delta = (\mu/\delta)\mu adt + (\mu/\delta)radt - (r(b + \mu\overline{s})/\delta)dt \;= \big((\mu/\delta)(\mu + r)a - (r(b + \mu\overline{s})/\delta)\big)dt,$$

since $\underline{P}(a) = \big(b + \mu(\overline{s} - a)\big)/\delta,$), $Eda = -\mu adt$. Consequently, $Ed\underline{P}(a) > r\underline{P}(a)dt$ if, and only if,

$$a > \tilde{a}^h = \frac{r(b+\mu\overline{s})}{(r+\mu)\mu} = \frac{r\hat{a}^h}{r+\mu} \ .$$

It can be shown, similarly to the proof in Appendix A that $\tilde{a}^h$ is the solution of equation $Z\big(\underline{P}(a)\big) = -\mu/\delta$. This implies $\underline{P}(\tilde{a}^h) = \overline{P}(\tilde{a}^l) = \tilde{p}$ and the chain of inequalities extending (25):

$$a^l < \tilde{a}^l < \tilde{a}^h < a^h,$$

As a result, $Ed\underline{P}(a) > r\underline{P}(a)dt$ for $a > a^h > \tilde{a}^h$ (and $a < \hat{a}^h$), because the difference $Ed\underline{P}(a) - r\underline{P}(a)dt$ is increasing in a. Inequality (27) is fulfilled for the off-equilibrium continuation of the EPF $p(a')$, where $a^h < a' = s + x/\mu < \hat{a}^h$ and $s > \overline{s}$, because $Edp(a') = rp(a')dt$.

Appendix C: *Solution of equation (30)*

One can represent a solution of (30) as the function series expansion for degrees of δ:

$$p(a) = p^{(0)}(a) + \delta p^{(1)}(a) + \delta^2 p^{(2)}(a) + \cdots \ . \qquad \text{(A1)}$$

Inserting this into (31) implies:

$$\tfrac{1}{2}(\sigma/\mu)^2\big(p^{(0)\prime\prime} + \delta p^{(1)\prime\prime} + \delta^2 p^{(2)\prime\prime} + \cdots\big) - b\big(p^{(0)\prime} + \delta p^{(1)\prime} + \delta^2 p^{(2)\prime} + \cdots\big) - r\big(p^{(0)} + \delta p^{(1)} + \delta^2 p^{(2)} + \cdots\big) = -\delta\big(p^{(0)\prime} + \delta p^{(1)\prime} + \delta^2 p^{(2)\prime} + \cdots\big)\big(p^{(0)} + \delta p^{(1)} + \delta^2 p^{(2)} + \cdots\big) \ .$$

Collecting terms with the same degrees of δ yields the system of interconnected equations:

$$\delta^0: \quad \tfrac{1}{2}(\sigma/\mu)^2 p^{(0)\prime\prime} - bp^{(0)\prime} - rp^{(0)} = 0 \qquad \text{(A2)}$$

$$\delta^1: \quad \tfrac{1}{2}(\sigma/\mu)^2 p^{(1)\prime\prime} - bp^{(1)\prime} - rp^{(1)} = -p^{(0)\prime}p^{(0)} \qquad \text{(A3)}$$

$$\delta^2: \quad \tfrac{1}{2}(\sigma/\mu)^2 p^{(2)\prime\prime} - bp^{(2)\prime} - rp^{(2)} = -p^{(0)\prime}p^{(1)} - p^{(1)\prime}p^{(0)}$$

…….

Equations corresponding to the degrees of δ higher than one are ruled out. Equation (A2) is the linear homogenous equation identical to (28) with the general solution $p^{(0)}(a) = A_1e^{\lambda_1 a} + A_2e^{\lambda_2 a}$. The derivative is $p^{(0)\prime}(a) = \lambda_1A_1e^{\lambda_1 a} + \lambda_2A_2e^{\lambda_2 a}$. Inserting this into the right-hand side of (A3) yields the non-homogenous linear equation:

$\frac{1}{2}(\sigma/\mu)^2 p^{(1)\prime\prime}(a) - bp^{(1)\prime}(a) - rp^{(1)}(a) =$

$$\lambda_1 A_1^2 e^{2\lambda_1 a} + (\lambda_1 + \lambda_2) A_1 A_2 e^{(\lambda_1+\lambda_2)a} + \lambda_2 A_2^2 e^{2\lambda_2 a} \,. \qquad \text{(A4)}$$

We will find a partial solution of this equation as

$$p^{(1)}(a) = B_1 e^{2\lambda_1 a} + B_{12} e^{(\lambda_1+\lambda_2)a} + B_2 e^{2\lambda_2 a} \,. \qquad \text{(A5)}$$

where B_1, B_{12}, B_2 are unknown parameters. Substituting this into (A4) and collecting terms with identical exponents yields equations on $B_i, i = 1{,}2$, and B_{12}, respectively;

$$2(\sigma/\mu)^2 \lambda_i^2 B_i e^{2\lambda_i a} - 2b\lambda_i B_i e^{2\lambda_i a} - rB_i e^{2\lambda_i a} = -\lambda_i A_i^2 e^{2\lambda_i a} \,,\ i = 1{,}2 \qquad \text{(A6)}$$

$$\frac{1}{2}(\sigma/\mu)^2 (\lambda_1 + \lambda_2)^2 B_{12} e^{(\lambda_1+\lambda_2)a} - b(\lambda_1 + \lambda_2) B_{12} e^{(\lambda_1+\lambda_2)a} - rB_{12} e^{(\lambda_1+\lambda_2)a} = (\lambda_1 + \lambda_2) A_1 A_2 e^{(\lambda_1+\lambda_2)a} \qquad \text{(A7)}$$

Rearrange (A6) to solve for B_i:

$$B_i (2(\sigma/\mu)^2 \lambda_i^2 - 2b\lambda_i - r) = -\lambda_i A_i^2 \,.$$

The left-hand side of this equation equals $B_i((3/2)(\sigma/\mu)^2 \lambda_i^2 - b\lambda_i)$, because λ_i satisfies the characteristic equation: $\frac{1}{2}(\sigma/\mu)^2 \lambda_i^2 - b\lambda_i - r = 0$. Consequently,

$$B_i = \frac{-\lambda_i A_i^2}{(3/2)(\sigma/\mu)^2 \lambda_i^2 - b\lambda_i} = \frac{-2A_i^2}{3(\sigma/\mu)^2 \lambda_i - 2b} \,.$$

This implies:

$$B_1 = \frac{-2A_1^2}{b + 3\sqrt{b^2 + 2(\sigma/\mu)^2 r}} \,, \quad B_2 = \frac{2A_2^2}{3\sqrt{b^2 + 2(\sigma/\mu)^2 r} - b} \,,$$

because

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 + 2(\sigma/\mu)^2 r}}{(\sigma/\mu)^2} \,.$$

Rearrange (A7) to solve for B_{12}:

$$B_{12}\left(\frac{1}{2}(\sigma/\mu)^2 (\lambda_1 + \lambda_2)^2 - b(\lambda_1 + \lambda_2) - r\right) = (\lambda_1 + \lambda_2) A_1 A_2 \,,$$

$$B_{12}\left(\frac{2b^2}{(\sigma/\mu)^2} - \frac{2b^2}{(\sigma/\mu)^2} - r\right) = \frac{2b}{(\sigma/\mu)^2} A_1 A_2 \,,$$

because $\lambda_1 + \lambda_2 = 2b/(\sigma/\mu)^2$. Hence,

$$B_{12} = -\frac{2bA_1A_2}{r(\sigma/\mu)^2} \ .$$

From (A1), (29), (A5), the solution of equation (30) is

$$p(a) \approx p^{(0)}(a) + \delta p^{(1)}(a) = A_1 e^{\lambda_1 a} + A_2 e^{\lambda_2 a} + \delta\left(B_1 e^{2\lambda_1 a} + B_{12} e^{(\lambda_1+\lambda_2)a} + B_2 e^{2\lambda_2 a}\right)$$

$$= A_1 e^{\lambda_1 a} + A_2 e^{\lambda_2 a}$$

$$+ \delta\left(\frac{-2A_1^2 e^{2\lambda_1 a}}{b + 3\sqrt{b^2 + 2(\sigma/\mu)^2 r}} - \frac{2bA_1A_2 e^{(\lambda_1+\lambda_2)a}}{r(\sigma/\mu)^2} + \frac{2A_2^2 e^{2\lambda_2 a}}{3\sqrt{b^2 + 2(\sigma/\mu)^2 r} - b}\right)$$

$$= A_1 e^{\lambda_1 a} + A_2 e^{\lambda_2 a} + \delta\left(\beta_1\left(A_1 e^{\lambda_1 a}\right)^2 + \beta_{12} A_1 A_2 e^{(\lambda_1+\lambda_2)a} + \beta_2\left(A_2 e^{\lambda_2 a}\right)^2\right).$$

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